Electromagnetic Waves in Conductors

Wave propagating through a vacuum or through insulating material such as glass or pure water } Charge density ρ_f and current density J_f

discussed in last section | Zero

In conductors → We do not independently control the flow of charge → and in general $J_f \neq 0$

According to the Ohm's law the (free) current density in a conductor → proportional to the electric field

$$J_f = \sigma E \quad \text{---(1)}$$

Maxwell's eq^{n}s for linear media

$$\left.\begin{aligned} &(i)\ \nabla\cdot E = \frac{1}{\varepsilon}\rho_f && (iii)\ \nabla\times E = -\frac{\partial B}{\partial t} \\ &(ii)\ \nabla\cdot B = 0 && (iv)\ \nabla\times B = \mu\sigma E + \mu\varepsilon\frac{\partial E}{\partial t} \end{aligned}\right\} \text{---(2)}$$

Continuity eq^n for free charge

$$\nabla\cdot J_f = -\frac{\partial \rho_f}{\partial t}, \quad \text{---(3)}$$

With Ohm's law and Gauss's law (i)

$$\frac{\partial \rho_f}{\partial t} = -\sigma(\nabla\cdot E) = -\frac{\sigma}{\varepsilon}\rho_f$$

$\Rightarrow$ for a homogeneous linear medium

$$\rho_f(t) = e^{-(\sigma/\varepsilon)t}\,\rho_f(0) \quad \text{---} (4)$$

$\Rightarrow$ Initial free charge density $\rho_f(0) \longrightarrow$ dissipates in a characteristic time $\tau \equiv \varepsilon/\sigma$.

$\hookrightarrow$ If you put some free charge on a conductor

$\downarrow$

it will flow out to the edges.

Time constant $\tau \longrightarrow$ measure of how "good" a conductor is

transient behaviour

Perfect conductor $\quad \sigma = \infty, \quad \tau = 0$

Good conductor $\quad \tau$ much less than the other relevant times in the problem ($\tau \ll \frac{1}{\omega}$)

Poor conductor $\quad \tau$ is greater than the characteristic time in the problem ($\tau \gg \frac{1}{\omega}$)

for $\rho_f = 0$, we have

$$\left.\begin{array}{ll} \text{(i)}\ \nabla\cdot E = 0 & \text{(iii)}\ \nabla\times E = -\dfrac{\partial B}{\partial t} \\ \text{(ii)}\ \nabla\cdot B = 0 & \text{(iv)}\ \nabla\times B = \mu\varepsilon\dfrac{\partial E}{\partial t} + \mu\sigma E \end{array}\right\} \text{--}(5)$$

applying the curl to (iii) and (iv)

We obtain modified wave equations for $\underline{E}$ and $\underline{B}$:

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$$\boxed{\nabla^2 E = \mu\varepsilon \frac{\partial^2 E}{\partial t^2} + \mu\sigma \frac{\partial E}{\partial t} \qquad \nabla^2 B = \mu\varepsilon \frac{\partial^2 B}{\partial t^2} + \mu\sigma \frac{\partial B}{\partial t}} \quad \text{--- (6)}$$

Plane-wave solutions,

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z-\omega t)}, \quad \tilde{B}(z,t) = \tilde{B}_0 e^{i(\tilde{k}z-\omega t)} \quad \text{--- (7)}$$

Here "wave number" $\tilde{k} \longrightarrow$ complex:

$$\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega \quad \text{--- (8)}$$

[put eq^n (7) in eq^n (6)]

$$\Rightarrow \quad \tilde{k} = k + i\kappa \quad \text{--- (9)}$$

where

$$\left.\begin{aligned} k &\equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2} \\ \kappa &\equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2} \end{aligned}\right\} \quad \text{--- (10)}$$

Imaginary part of $\tilde{k} \longrightarrow$ attenuation of the wave

$\downarrow$

decreasing amplitude with z

$$\tilde{E}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}, \quad \tilde{B}(z,t) = \tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)} \quad \text{--- (11)}$$

Skin depth $\longrightarrow$ distance it takes to reduce the amplitude by a factor of $\frac{1}{e}$

Measure of how far the wave penetrates in the conductor $\quad d \equiv \frac{1}{\kappa}$; --- (12)

Real part of $\tilde{k}$ $\rightarrow$ determines the wavelength, propagation speed and index of refraction

$$\lambda = \frac{2\pi}{k}, \quad v = \frac{\omega}{k}, \quad n = \frac{ck}{\omega} \quad \text{——} (13)$$

attenuated waves, eq^n (11), satisfy the wave eq^n (6) for any $\tilde{E}_0$ and $\tilde{B}_0$.

Maxwell's eq^n (5) $\longrightarrow$ impose further constraints

help to find relative amplitudes, phases and polarizations of E and B.

(i) and (ii) rule out any z components : the fields are transverse.

We may orient our axes so that E is polarized along the x direction:

$$\tilde{E}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{x}. \quad \text{—} (14)$$

(iii) gives

$$\tilde{B}(z,t) = \frac{\tilde{k}}{\omega} \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{y}. \quad \text{—} (15)$$

Eq. (iv) $\rightarrow$ same thing

$\Rightarrow$ Electric and magnetic fields are mutually perpendicular

$\tilde{K} \rightarrow$ could be expressed in terms of its modulus & phase:

$$\tilde{K} = K e^{i\phi} \quad \text{--- (16)}$$

Where $K \equiv |\tilde{k}| = \sqrt{k^2 + K^2} = \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}$ --- (17)

and $\phi \equiv \tan^{-1}(K/k)$ --- (18)

in eq^n (14) & (15), the complex amplitudes $\tilde{E}_0 = E_0 e^{i\delta_E}$ and $\tilde{B}_0 = B_0 e^{i\delta_B}$ are related

$$B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E} \quad \text{--- (18)}$$

Evidently the electric & magnetic fields are no longer in phase

$$\delta_B - \delta_E = \phi \quad \text{--- (19)}$$

The (real) amplitudes of E and B are related by

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \quad \text{--- (20)}$$

The (real) electric & magnetic fields are, finally

$$\left.\begin{aligned} E(z,t) &= E_0 e^{-kz} \cos(kz - \omega t + \delta_E)\,\hat{x} \\ B(z,t) &= B_0 e^{-kz} \cos(kz - \omega t + \delta_E + \phi)\,\hat{y} \end{aligned}\right\} \quad \text{--- (21)}$$